# SECTION 4.1: MAXIMA AND MINIMA (EXTREMA)

**Name:** ____________________________________________

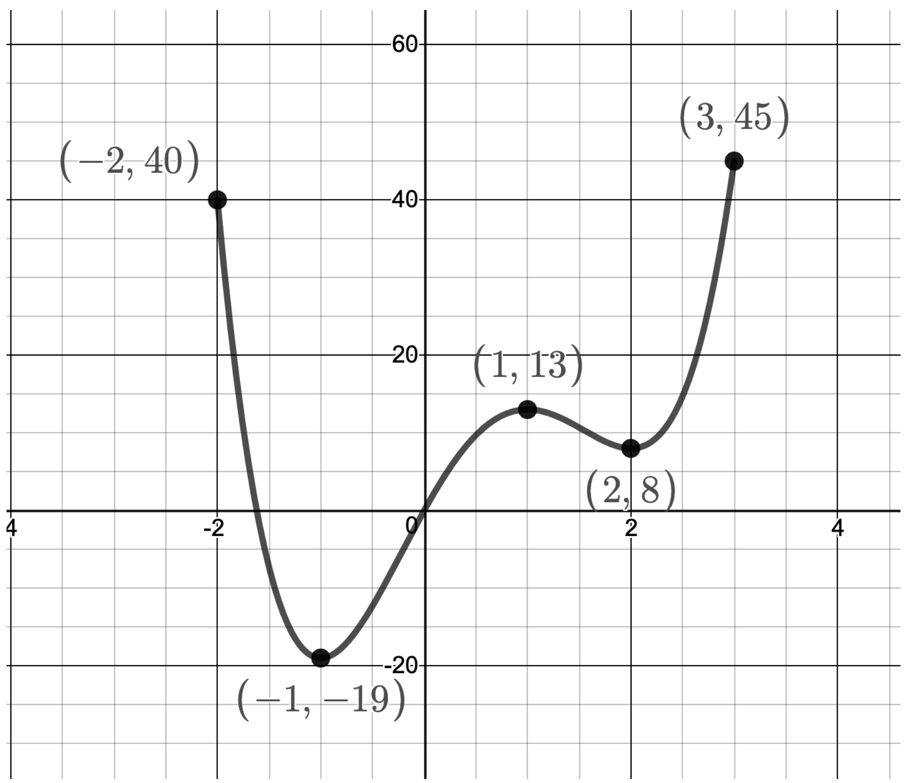
**DEFINITIONS:** Suppose $f$ is defined on an interval $I$.

- The value $f(c)$ is called the **absolute (or global) maximum** of $f$ on $I$ if $f(c) \geq f(x)$ for all $x$ in $I$.
- The value $f(c)$ is called the **absolute (or global) minimum** of $f$ on $I$ if $f(c) \leq f(x)$ for all $x$ in $I$.
- The value $f(c)$ is called a **local maximum** if $f(c) \geq f(x)$ for all $x$ in some open interval in $I$ containing $c$.
- The value $f(c)$ is called a **local minimum** if $f(c) \leq f(x)$ for all $x$ in some open interval in $I$ containing $c$.

**NOTE:** The maximum and minimum values of a function are called the '**extreme values**' or the '**extrema**.'

Geometrically, the absolute extrema of function correspond to the highest and lowest points on the graph, respectively. The 'local' extrema of a function correspond to points which are the highest or lowest points when we zoom in near a point and compare function values on either side of that point.

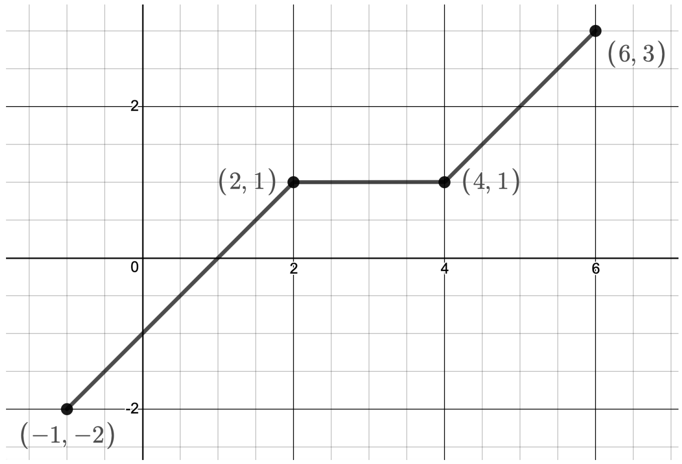
**EXAMPLE 1:** The graph of $y = f(x)$ is given below. List the absolute and local extrema.




The absolute maximum of $f$ is 45 at $x = 3$ and the absolute minimum of $f$ is $-19$ at $x = -1$.

There is a local maximum at $(1, 13)$ and local minimums at $(-1, -19)$ and $(2, 8)$

**EXAMPLE 2:** The graph of $y = f(x)$ is given below. List the absolute and local extrema.

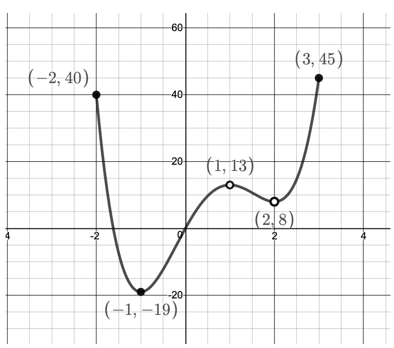



The absolute maximum of $f$ is 3 at $x = 6$ and the absolute minimum of $f$ is $-2$ at $x = -1$.

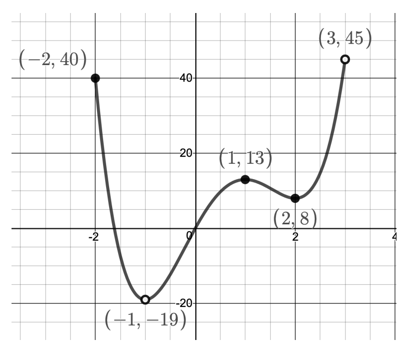
Because of the '=' in '$\geq$' and '$\leq$' in the definitions of local maximum and minimum, respectively, we have that the points $(x, 1)$ for $2 \leq x < 4$ are local maximums and, likewise, points $(x, 1)$ for $2 < x \leq 4$ are local minimums.

**EXTREME VALUE THEOREM:** (EVT) If $f$ is **continuous** on $[a, b]$, then $f$ attains its extrema on $[a, b]$. That is, there are values $x_1$ and $x_2$ in $[a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x$ in $[a, b]$.

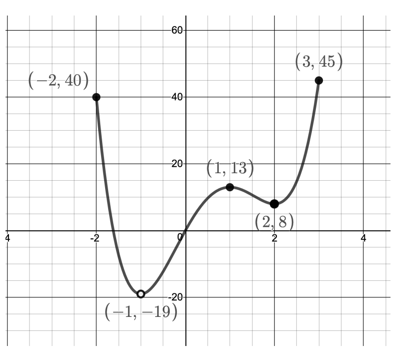
Note that the EVT is phrased in terms of a conditional 'if . . . then' statement. That is, to be **guaranteed** $f$ has absolute extrema over the interval $[a, b]$, $f$ needs to be continuous. If $f$ is not continuous, $f$ may or may not have absolute extrema as the following graphs illustrate.



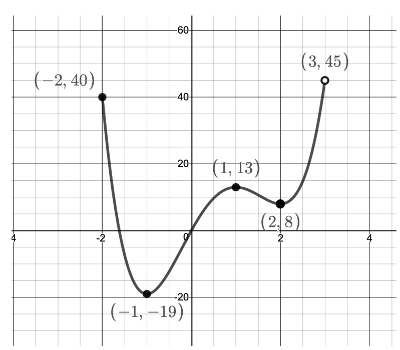

$f$ has both an absolute max and min.




$f$ has neither an absolute max nor min.




$f$ has an absolute max but not min.




$f$ has an absolute min but not max.

The extreme values of a continuous function on an interval $[a, b]$ could occur at the endpoints, $a$ and/or $b$, or in the interior, $(a, b)$. If the latter, the extrema are local extrema. The next theorem helps us locate local extrema.

**FERMAT'S THEOREM:** If $f(c)$ is a local extreme value of $f$, then $f'(c) = 0$ or $f'(c)$ does not exist.

**PROOF:** Suppose $f$ has a local maximum at $(c, f(c))$. If $f'(c)$ does not exist, then we are done as this is provided for in the theorem. Hence, what we really need to show is that if $f'(c)$ does exist, then $f'(c) = 0$.

Since $(c, f(c))$ is a local maximum, then for all $x$ near $c$, $f(x) \leq f(c)$. Recall the definition of derivative:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Also note that for this limit to exist,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

For $x$ near $c$ but $x < c$, $x - c < 0$. Since $f(x) \leq f(c)$, $f(x) - f(c) \leq 0$ so $\dfrac{f(x) - f(c)}{x - c} \geq 0$. Hence,

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

For $x$ near $c$ but $x > c$, $x - c > 0$. Since $f(x) - f(c) \leq 0$, however, this means $\dfrac{f(x) - f(c)}{x - c} \leq 0$. Hence,

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

Since $f'(c)$ exists, $0 \leq \displaystyle\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$. Hence, $f'(c) = 0$.

We leave it to the interested reader to adjust the argument above for the case $f$ has a local minimum at $(c, f(c))$.

**NOTE:** Fermat's Theorem is another 'if ... then' construction. There are examples where $f'(c) = 0$ or $f'(c)$ does not exist and $f(c)$ is **NOT** a local extreme value. For instance, take $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ at $x = 0$.

**DEFINITION:** A value $c$ in the domain of $f$ is called a **critical number** if $f'(c) = 0$ or $f'(c)$ does not exist.

Putting all our work together, we have all the pieces necessary to optimize continuous functions on closed intervals:

## OPTIMIZING CONTINUOUS FUNCTIONS ON CLOSED BOUNDED INTERVALS:

1. Verify $f$ is continuous on $[a, b]$, so the EVT applies.
2. Find the critical numbers of $f$: solve $f'(x) = 0$ or determine where $f'(x)$ does not exist.
3. Evaluate $f$ at the endpoints and the critical values:
    - the largest function value is the absolute maximum of $f$.
    - the smallest function value is the absolute minimum of $f$.

**EXAMPLE 3:** Find the absolute extrema of $f(x) = 4x^3 + 15x^2 - 18x + 4$ on $[0, 1]$.

1. Find and simplify $f'(x)$.

   Ans: $f'(x) = 12x^2 + 30x - 18$

2. Find the critical numbers of $f$ by solving $f'(x) = 0$. Record the solutions which lie in the interval $(0, 1)$.

   Ans: $f'(x) = 12x^2 + 30x - 18 = 0$ when $x = -3$ or $x = \frac{1}{2}$. Only $x = \frac{1}{2}$ lies in the interval $(0, 1)$.

3. Evaluate $f$(critical numbers), $f(0)$, and $f(1)$. Record the absolute maximum and absolute minimum.

   Ans: $f(0) = 4$, $f\left(\frac{1}{2}\right) = -\frac{3}{4}$ and $f(1) = 5$. The absolute max is 5 and the absolute min is $-\frac{3}{4}$.

**EXAMPLE 4: (VIDEO)** Find the absolute extrema of $f(x) = \dfrac{4x}{x^2 + 1}$ on $[-2, 0]$.

1. Find and simplify $f'(x)$.

   Ans: $f'(x) = \dfrac{4 - 4x^2}{(x^2 + 1)^2}$

2. Find the critical numbers of $f$ by solving $f'(x) = 0$. Record the solutions which lie in the interval $(-2, 0)$.

   Ans: $f'(x) = \dfrac{4 - 4x^2}{(x^2 + 1)^2} = 0$. When $4 - 4x^2 = 0$ so $x = \pm 1$. Only $x = -1$ is in the interval $(-2, 0)$.

3. Evaluate $f$(critical numbers), $f(-2)$, and $f(0)$. Record the absolute maximum and absolute minimum.

   Ans: $f(-2) = -\frac{8}{5}$, $f(-1) = -2$, and $f(0) = 0$. The absolute max is 0 and the absolute min is $-2$.

**EXAMPLE 5: (VIDEO)** Find the absolute extrema of $f(x) = 2\cos(x) - \cos(2x) + 1$ on $[0, 2\pi]$.

1. Find and simplify $f'(x)$.

   Ans: $f'(x) = -2\sin(x) + 2\sin(2x)$

2. Find the critical numbers of $f$ by solving $f'(x) = 0$. Record the solutions which lie in the interval $(0, 2\pi)$.

   **RECALL:** $\sin(2x) = 2\sin(x)\cos(x)$

   Ans: $f'(x) = -2\sin(x) + 2\sin(2x) = -2\sin(x) + 4\sin(x)\cos(x) = 0$ when $x = \frac{\pi}{3}$, $x = \pi$, and $x = \frac{5\pi}{3}$.

3. Evaluate $f$(critical numbers), $f(0)$, and $f(2\pi)$. Record the absolute maximum and absolute minimum.

   Ans: $f(0) = f(2\pi) = 2$, $f(\pi) = -2$, $f\left(\frac{\pi}{3}\right) = f\left(\frac{5\pi}{3}\right) = \frac{5}{2}$.

   The absolute max is $\frac{5}{2}$ and the absolute min is $-2$.

**EXAMPLE 6: (VIDEO)** Assume $a > 0$ is a constant. Find the absolute extrema of $f(x) = x\sqrt[3]{x-a}$ on $[0, 3a]$.

1. Find and simplify $f'(x)$.

   Ans: $f'(x) = \dfrac{4x-3a}{3(x-a)^{2/3}}$

2. Find the critical numbers of $f$ by solving $f'(x) = 0$ or where $f'(x)$ does not exist. Record the solutions which lie in the interval $(0, 3a)$.

   Ans: $f'(x) = \dfrac{4x-3a}{3(x-a)^{2/3}}$ does not exist when $3(x-a)^{2/3} = 0$ so when $x = a$.

   $f'(x) = \dfrac{4x-3a}{3(x-a)^{2/3}} = 0$ when $4x - 3a = 0$ so when $x = \frac{3a}{4}$.

3. Evaluate $f$(critical numbers), $f(0)$, and $f(3a)$. Record the absolute maximum and absolute minimum.

   Ans: $f(0) = 0$, $f\left(\frac{3a}{4}\right) = -\dfrac{3a^{4/3}}{4^{4/3}}$, $f(a) = 0$, $f(3a) = 3a^{4/3}2^{1/3}$.

   The absolute max is $3a^{4/3}2^{1/3}$ and the absolute min is $-\dfrac{3a^{4/3}}{4^{4/3}}$.

**HOMEWORK:** Section 4.1: 1 - 81 every other odd.